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Nonlinear supersymmetry: from classical to quantum mechanics

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Abstract

Quantization of the nonlinear supersymmetry faces a problem of a quantum anomaly. For some classes of superpotentials, the integrals of motion admit corrections guaranteeing the preservation of the nonlinear supersymmetry at the quantum level. With an example of a system realizing the nonlinear superconformal symmetry, we discuss the nature of such corrections and speculate on their possible general origin.

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1. Introduction

Nonlinear symmetry is characterized by the polynomial commutation relations of the integrals of motion [1]. Nonlinear generalization of the usual supersymmetry [2] was obtained originally by employing the higher derivative supercharges [3]. The higher derivative realization of the nonlinear supersymmetry has been studied in various aspects, see [3–9].

Another generalization of supersymmetric quantum mechanics is a construction of the minimally bosonized supersymmetry [10], in which a reflection operator is employed for the role of the grading operator. The development of the construction in application to the study of hidden symmetries of the purely parabosonic systems resulted in the observation of nonlinear supersymmetry independent of the original line of research [11].

A new wave of interest in nonlinear supersymmetry [12–20] was triggered by the observation of its close relation to the quasi-exactly solvable systems [21]. The relationship was revealed, in particular, under the study of the quantum anomaly problem [12] arising in the quantization of classical systems possessing nonlinear supersymmetry [11].

In the present paper, we discuss the solution of the quantum anomaly problem associated with the nonlinear supersymmetry, emphasizing the analysis of the nature of the corresponding quantum corrections.

The layout of the paper is as follows. In section 2, the origin of the quantum anomaly problem and its solution resulting in the holomorphic form of the nonlinear supersymmetry are discussed together with a universal algebraic structure underlying the latter. In section 3 we consider a particular case of the second-order supersymmetry given by the Calogero-like quantum Hamiltonian. Section 4 is devoted to the discussion of the nonlinear generalization of the superconformal symmetry and its interpretation from the point of view of the special reduction procedure applied to the associated free particle system. In section 5 we specify some problems to be interesting for further consideration.

2. Quantum anomaly and nonlinear holomorphic supersymmetry

Let us consider a system described by the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 - W^2(x)) - gW'(x)\theta^+\theta^- + \mathrm{i}\theta^+\dot{\theta}^- \tag{2.1}$$

with Grassmann variables $\theta^\pm$, $(\theta^+)^* = \theta^-$, a superpotential $W(x)$ and a 'boson–fermion coupling constant' g. At $|g| = 1$ the pseudo-classical system (2.1) underlies Witten supersymmetric quantum mechanics with associated Lie superalgebra of the integrals of motion, while at $|g| = n, n = 2, 3, \ldots,$ it possesses the nonlinear supersymmetry [11, 12]. This can easily be seen in the canonical approach where the Hamiltonian

$$H = \tfrac{1}{2}(p^2 + W^2(x)) + nW'(x)\theta^+\theta^- \qquad n \in \mathbb{Z} \tag{2.2}$$

and the Poisson–Dirac brackets $\{x, p\} = 1, \{\theta^+, \theta^-\} = -\mathrm{i}$ correspond to Lagrangian (2.1) with $g = n$. Due to the invariance of the Lagrangian under $g \to -g, W(x) \to -W(x)$, we shall imply $n \in \mathbb{Z}_+$ in what follows. The system (2.2) possesses the two odd integrals of motion

$$Q^+ = z^n\theta^+ \qquad Q^- = (Q^+)^* = \bar{z}^n\theta^- \tag{2.3}$$

in addition to the Hamiltonian which in terms of the complex variable

$$z = W(x) + \mathrm{i}p \tag{2.4}$$

is represented in the form

$$H = \tfrac{1}{2}(z\bar{z} + \mathrm{i}n\{z, \bar{z}\}\theta^+\theta^-). \tag{2.5}$$

For integer values of the coupling constant, $g = n$, the projections on the unit of the Grassmann algebra of the instant frequencies of the oscillator-like bosonic, z $(\bar{z})$, and fermionic, θ^+ (θ^-), variables are commensurable (being equal to $-W'(x)$ and $nW'(x)$ for z and θ^+, respectively), which guarantees the existence of the local in time odd integrals of motion (2.3) [11, 12].

The integrals satisfy the relations

$$\{Q^+, Q^-\} = -\mathrm{i}H^n \qquad \{Q^+, Q^+\} = \{Q^-, Q^-\} = 0 \qquad \{Q^\pm, H\} = 0. \tag{2.6}$$

The case $n = 0$ is characterized by the trivial odd integrals of motion, $Q^\pm = \theta^\pm$. The case $n = 1$ corresponds to the linear supersymmetry, for which the supercharges and the Hamiltonian form a classical Lie superalgebra corresponding to Witten supersymmetric quantum mechanics. For $n = 2, 3, \ldots$ the system possesses the nonlinear supersymmetry characterized by the nonlinear (polynomial of the order n) Poisson superalgebra of the integrals of motion.

The essential difference of the nonlinear supersymmetry from the linear one appears as soon as we try to quantize the system. Taking the direct quantum analogues for the supercharges,

$$\hat{Q}^+ = (\hat{Q}^-)^\dagger = \hat{z}^n\hat{\theta}^+ \qquad \text{with} \quad \hat{z} = W(x) + \hbar\frac{\mathrm{d}}{\mathrm{d}x} \quad \hat{\theta}^+ = \sqrt{\hbar}\sigma_+ \tag{2.7}$$

where $\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2)$, and for the Hamiltonian,

$$\hat{H} = \frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + W^2(x) + n\hbar W'(x)\sigma_3\right) \tag{2.8}$$

one finds that at $n = 1$ the quantum supercharges (2.7) are the integrals of motion, $[\hat{Q}^{\pm}, \hat{H}] = 0$, for any choice of the superpotential $W(x)$, and the superalgebra they form together with the Hamiltonian is a direct quantum analogue of (2.6) with $n = 1$. However, for $n \geqslant 2$, the odd operators (2.7) commute with the Hamiltonian (2.8) only for the quadratic superpotential

$$W(x) = w_2x^2 + w_1x + w_0. \tag{2.9}$$

In this case the classical nonlinear superalgebra (2.6) changes its form for $[\hat{Q}^+, \hat{Q}^-]_+ = \hat{H}^n + P_{n-1}(\hat{H})$, where $P_{n-1}(\hat{H})$ is a polynomial of the order $n-1$ with the coefficients given in terms of $w_1^2 - 4w_0w_2$ and w_2^2, and disappearing for $\hbar \to 0$ [12]. For the superpotential of a generic form, $[\hat{Q}^{\pm}, \hat{H}] \neq 0$, and we face the problem of the quantum anomaly.

In order to resolve this problem, one could look for the class of the quantum analogues of the classical supercharges (2.3) whose conservation would be consistent with the quantum Hamiltonian of the fixed form (2.8). Then, one of the possibilities is to suppose that the supercharge $\hat{Q}^+$ has a holomorphic form depending only on $\hat{z}$ (and not depending on $\hat{\bar{z}}$), being a polynomial of the order n in it. The condition of conservation for such a supercharge is reduced to the equation

$$n(n^2-1)\hbar\frac{d}{dx}(\hbar^2W'' - \omega^2W) = 0 \tag{2.10}$$

where ω^2 is a constant. For $\omega \neq 0$, the general solution to equation (2.10) gives a three-parametric class of solutions

$$W(x) = w_+ e^{\omega x} + w_- e^{-\omega x} + w_0 \tag{2.11}$$

while for $\omega = 0$ the solution is (2.9). Different choices for the parameters w, $w_\pm$ and w_0 give rise then to a broad class of quasi-exactly solvable and exactly solvable quantum mechanical systems [12].

For a quasi-exactly solvable Hamiltonian only a finite number of the eigenstates and eigenvalues can be found in a purely algebraic manner [21]. A finite-dimensional subspace spanned by such eigenstates provides a finite-dimensional representation of the $sl(2, R)$ algebra, and the dimension of the representation is determined by a natural parameter appearing explicitly in the Hamiltonian. In the case of the nonlinear supersymmetry of the order n, the zero mode subspaces of supercharges can be associated with the n-dimensional representations of the $sl(2, R)$ [12]. In particular, the systems with nonlinear supersymmetry corresponding to the quadratic superpotential (2.9) can be related [16] to the quasi-exactly solvable systems with quartic potential appearing in the context of the PT-symmetric quantum mechanics [22].

The described class of systems with nonlinear supersymmetry given by the holomorphic supercharges corresponding to the superpotentials (2.11) and (2.9) admits the following generalization. First, we note that $[\hat{z}, \hat{\bar{z}}] = 2\hbar W'(x)$ and $[\hat{z}, \hat{\bar{z}}]_+ = 2\big(W^2(x) - \hbar^2\frac{d^2}{dx^2}\big)$. Then, generalizing the operators $\hat{z}$ and $\hat{\bar{z}}$ defined by equation (2.7) for the pair of mutually conjugate operators of arbitrary nature, Z and $\bar{Z} = Z^\dagger$, we can change the Hamiltonian and equation (2.10) for

$$\mathcal{H}_n = \tfrac{1}{4}([Z, \bar{Z}]_+ + n[Z, \bar{Z}]\sigma_3) \tag{2.12}$$

$$[Z, [Z, [Z, \bar{Z}]]] = \omega^2[Z, \bar{Z}] \qquad [\bar{Z}, [\bar{Z}, [Z, \bar{Z}]]] = \bar{\omega}^2[Z, \bar{Z}] \tag{2.13}$$

where $\bar{\omega} = \omega^*$. Relations (2.12) guarantee the existence of the two integrals of motion for the system (2.12),

$$\mathcal{Q}_n^+ = \prod_{k=0}^{n-1}\left(Z + \left(k - \frac{n-1}{2}\right)\omega\right)\cdot\sigma_+ \qquad \mathcal{Q}_n^- = (\mathcal{Q}^+)^\dagger \tag{2.14}$$

whose structure is corrected in comparison with the classical case by a trivial shift of bosonic factors.

Nonlinear algebraic relations similar to relations (2.13) also appeared in the context of spin integrable systems under the name of Dolan–Grady relations [23]. Let us stress that here the Dolan–Grady relations arise as necessary conditions for anomaly-free quantization of pseudo-classical systems with nonlinear holomorphic supersymmetry.

Through these relations, the Onsager algebra can be generated [16]. The case $\omega = 0$ corresponds to a more simple form of the contracted Onsager algebra generated recursively by $Z_0 \equiv Z$ and $\bar{Z}_0 \equiv \bar{Z}$ via the contracted ($\omega = 0$) Dolan–Grady relations:

$$\begin{aligned} &[Z_m, \bar{Z}_n] = B_{m+n+1} \qquad &&[Z_n, B_m] = Z_{m+n} \qquad &&[B_m, \bar{Z}_n] = \bar{Z}_{m+n} \\ &[Z_m, Z_n] = 0 &&[\bar{Z}_m, \bar{Z}_n] = 0 &&[B_m, B_n] = 0 \end{aligned} \tag{2.15}$$

where $m, n \in \mathbb{Z}_+$ and $B_0 = 0$ is implied. In the general case of $\omega \neq 0$, the Onsager algebra has a similar but more complicated form, see [16]. In both cases of $\omega = 0$ and $\omega \neq 0$, the Onsager algebra admits an infinite set of mutually commuting quadratic operators,

$$J_\lambda^l = \frac{1}{2}\sum_{p=1}^{l}([\bar{Z}_{p-1}, Z_{l-p}]_+ - B_l B_{l-p}) - \frac{\lambda}{2}B_l$$

where $l = 1, 2, \ldots,$ and λ is a parameter. Putting $\lambda = n \in \mathbb{N}$ and introducing the operators $\mathcal{J}_n^l = J_n^l\sigma_-\sigma_+ + J_{-n}^l\sigma_+\sigma_-$, $\sigma_- = \sigma_+^\dagger$, one finds that the Hamiltonian (2.12) belongs to the infinite set of mutually commuting even operators $\mathcal{J}_n^l$, $l = 1, 2, \ldots, \mathcal{J}_n^1 = \mathcal{H}_n$, to be quadratic in the generators of the Onsager algebra, and that the nonlinear superalgebra takes the form

$$\left[\mathcal{Q}_n^+, \mathcal{Q}_n^-\right]_+ = \mathcal{H}_n^n + P_{n-1}\left(\mathcal{J}_n^l\right) \qquad \left[\mathcal{Q}^\pm, \mathcal{J}_n^l\right] = \left[\mathcal{J}_n^l, \mathcal{J}_n^k\right] = 0 \tag{2.16}$$

where $P_{n-1}\left(\mathcal{J}_n^l\right)$ is a polynomial of the order $n - 1$ [16]. For systems with a finite number of degrees of freedom, there is only a finite number of independent integrals $\mathcal{J}_n^l$.

3. Second-order supersymmetric quantum mechanics

Though a universal algebraic structure associated with Dolan–Grady relations allows one to realize nonlinear holomorphic supersymmetry in nontrivial (Riemann) and noncommutative geometries [17] as well as to generalize the construction for the case of nonlinear pseudo-supersymmetry [16] related to the PT-symmetric quantum mechanics [24], it generates a rather special class of systems possessing nonlinear supersymmetry. But since there is no one-to-one correspondence between the classical canonical and the quantum unitary transformations, one can look for other forms of nonlinear supersymmetry proceeding from the classical representations different from the holomorphic one. For instance, one could look for the classical formulation characterized by the supercharges being polynomials of the nth degree in the momentum p. The problem of finding such a formulation can be solved completely in the simplest case $n = 2$ [12]. For it, the Hamiltonian and the supercharges are fixed as

$$H = \frac{1}{2}\left(p^2 + W^2(x) - \frac{\nu}{W^2(x)}\right) + 2W'(x)\theta^+\theta^- \qquad \nu \in \mathbb{R} \tag{3.1}$$

$$Q^{\pm} = \frac{1}{2}\left((\pm \mathrm{i}p + W(x))^2 + \frac{\nu}{W^2(x)}\right)\theta^{\pm}. \tag{3.2}$$

The system (3.1), (3.2) is characterized by the order $n = 2$ superalgebra $\{Q^+, Q^-\} = -\mathrm{i}(H^2 + \nu)$. Note that for $W(x) = x$, the part of (3.1) without the last nilpotent term takes the form of the Hamiltonian for the two-body Calogero problem. The peculiarity of the Calogero-like $n = 2$ supersymmetric system (3.1), (3.2) in comparison with the nonlinear holomorphic supersymmetry is that it admits the quantum anomaly-free formulation for the superpotential $W(x)$ of an arbitrary form. The anomaly-free quantum version of the $n = 2$ supersymmetric system (3.1), (3.2) is given by the operators

$$\hat{H} = \frac{1}{2}\left(-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + W^2 - \frac{\nu}{W^2} + 2\hbar W'\sigma_3 + \Delta(W)\right) \tag{3.3}$$

$$\hat{Q}^+ = (\hat{Q}^-)^{\dagger} = \frac{1}{2}\left(\left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + W\right)^2 + \frac{\nu}{W^2} - \Delta(W)\right)\hat{\theta}^+ \tag{3.4}$$

with the quantum correction

$$\Delta(W) = \frac{\hbar^2}{4W^2}(2W''W - W'^2) = \hbar^2\frac{1}{\sqrt{W}}(\sqrt{W})''. \tag{3.5}$$

Note that here the quadratic and exponential forms of the superpotential also play a special role: for $W(x) = (w_1x + w_0)^2$, $\Delta(W) = 0$, while for $W(x) = (w_+\,\mathrm{e}^{\omega x} + w_-\,\mathrm{e}^{-\omega x})^2$, $\Delta(W) = \hbar^2\omega^2 = \text{const}$.

The inclusion of the quantum correction in the Hamiltonian and supercharges is crucial for the preservation of the nonlinear supersymmetry: without it the supercharges would not be the quantum integrals of motion. The quantum integrals (3.3) and (3.4) satisfy the relation to be the direct quantum analogue of the corresponding classical relation: $[\hat{Q}^+, \hat{Q}^-]_+ = \hat{H}^2 + \nu$.

The quantum systems (3.3), (3.4) were the first ones discussed in the context of the nonlinear ($n = 2$) supersymmetry [3]. But it is interesting to note that a Hamiltonian structure similar to (3.3) was discussed earlier in the context of partial symmetry breaking in $N = 4$ supersymmetric quantum mechanics [25], and that the quantum term (3.5) appeared also in the method of constructing new solvable potentials via the operator transformations, see [2] and references therein. The order $n = 2$ nonlinear supersymmetry given by equations (3.3), (3.4) with no restrictions on the form of the superpotential admits a generalization for $n > 2$ for a class of superpotentials of a special form related to some quasi-exactly solvable systems [14].

4. Nonlinear superconformal symmetry and reduction

The natural question arising here is whether the origin of the quantum corrections appearing under construction of nonlinear supersymmetry can be explained to some extent. To clarify this point, we consider a class of the systems [26, 27] possessing nonlinear supersymmetry and related to the superconformal mechanics model [28]. As we shall see, in the simplest case $n = 2$, the corresponding quantum system is a particular case of the $n = 2$ supersymmetric systems (3.3), (3.4), and the corresponding quantum corrections necessary for conservation of the nonlinear supersymmetry appear for it in a natural way via a special reduction procedure applied to the associated free particle spinning system.

So, let us consider a classical system given by the Hamiltonian (2.2) with the superpotential $W(x) = \alpha x^{-1}$:

$$H = \tfrac{1}{2}(p^2 + \alpha(\alpha - n\theta^+\theta^-)x^{-2}). \tag{4.1}$$

At $n = 1$, equation (4.1) is the Hamiltonian of the superconformal mechanics model [28], which possesses a broader superconformal symmetry generated by the Hamiltonian and the three even integrals,

$$D = \tfrac{1}{2}xp - tH \qquad K = \tfrac{1}{2}x^2 - 2tD - t^2H \qquad \Sigma = \theta^+\theta^- \tag{4.2}$$

as well as by the four odd integrals of motion

$$Q_a = p\theta_a + \alpha x^{-1}\epsilon_{ab}\theta_b \qquad S_a = p\theta_a - tQ_a$$

where $a, b = 1, 2$ and $\theta^\pm = \frac{1}{\sqrt{2}}(\theta_1 \pm \mathrm{i}\theta_2)$. All these integrals satisfy the conservation equation of the form $\frac{\mathrm{d}}{\mathrm{d}t}I = \frac{\partial}{\partial t}I + \{I, H\} = 0$. In the case of $n = 2, 3, \ldots$, the set of even integrals of motion is the same as in the superconformal mechanics model ($n = 1$) with the correspondingly changed boson–fermion coupling constant in the Hamiltonian, but instead of the four odd integrals of motion, there are $2(n + 1)$ supercharges,

$$S^+_{n,l} = (x + \mathrm{i}tz)^l z^{n-l}\theta^+ \qquad S^-_{n,l} = \left(S^+_{n,l}\right)^* \qquad l = 0, 1, \ldots, n \tag{4.3}$$

where the even complex variable z is defined by equation (2.4) with $W(x) = \alpha x^{-1}$. These integrals form the following nonlinear superconformal algebra $osp(2|2)_n$ [26, 27]:

$$\{D, H\} = H \qquad \{D, K\} = -K \qquad \{K, H\} = 2D \tag{4.4}$$

$$\{D, S^\pm\} = \left(\frac{n}{2} - l\right) S^\pm_{n,l} \qquad \left\{\Sigma, S^\pm_{n,l}\right\} = \mp\mathrm{i}S^\pm_{n,l} \tag{4.5}$$

$$\left\{H, S^\pm_{n,l}\right\} = \pm\mathrm{i}lS^\pm_{n,l-1} \qquad \left\{K, S^\pm_{n,l}\right\} = \pm\mathrm{i}(n - l)S^\pm_{n,l+1} \tag{4.6}$$

$$\left\{S^+_{n,m}, S^-_{n,l}\right\} = -\mathrm{i}(2H)^{n-m-1}(2K)^{l-1}\alpha_D^{m-l}\left[4HK - \mathrm{i}\Sigma(n(m - l)\alpha_D + 4\alpha l(n - m))\right] \tag{4.7}$$

where in the last relation $\alpha_D \equiv \alpha - 2\mathrm{i}D$ and $m \geqslant l$, and the brackets between the odd integrals for $m < l$ are obtained from it by the complex conjugation. The nonlinear superconformal algebra $osp(2|2)_n$ contains the bosonic Lie subalgebra $so(1, 2) \oplus u(1)$ generated by the even integrals, while the set of odd supercharges constitutes the two spin-$\frac{n}{2}$ representations of the bosonic subalgebra. At $n = 1$ the nonlinear superconformal algebra is reduced to the Lie superalgebra $osp(2|2)$.

The nonlinear superconformal symmetry admits the anomaly-free quantization. The corresponding quantum corrections to the integrals of motion can be found directly giving the following quantum analogues of the $osp(2|2)_n$ generators:

$$\hat{H} = \frac{1}{2}\left(-\hbar^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (a_n + b_n\hbar\sigma_3)\frac{1}{x^2}\right) \tag{4.8}$$

$$\hat{D} = -\frac{\mathrm{i}}{2}\hbar\left(x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{1}{2}\right) - \hat{H}t \qquad \hat{K} = \frac{1}{2}x^2 - 2\hat{D}t - \hat{H}t^2 \qquad \hat{\Sigma} = \frac{\hbar}{2}\sigma_3 \tag{4.9}$$

$$\hat{S}^+_{n,l} = (x + \mathrm{i}t\mathcal{D}_{\alpha-n+1})(x + \mathrm{i}t\mathcal{D}_{\alpha-n+2})\cdots(x + \mathrm{i}t\mathcal{D}_{\alpha-n+l})\mathcal{D}_{\alpha-n+l+1}\cdots\mathcal{D}_{\alpha}\sigma_+ \qquad \hat{S}^-_{n,l} = \left(\hat{S}^+_{n,l}\right)^\dagger \tag{4.10}$$

where

$$\begin{aligned} &a_n = \alpha_n^2 + \frac{\hbar^2}{4}(n^2-1) \qquad b_n = -n\alpha_n \qquad \alpha_n = \alpha - \frac{\hbar}{2}(n-1) \\ &\mathcal{D}_{\alpha-k} = \left(\hbar\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\alpha}{x}\right) - \hbar\frac{k}{x}. \end{aligned} \tag{4.11}$$

The second terms in a_n, α_n and $\mathcal{D}_{\alpha-k}$ are the quantum corrections which guarantee the conservation of the quantum analogues of the integrals of motion (4.1), (4.2) and (4.3). We note that since the not depending explicitly on time supercharges $S^+_{n,0} = z^n\theta^+$ and $S^-_{n,0} = \left(S^+_{n,0}\right)^*$ have the holomorphic form and the Hamiltonian (4.1) can be represented in the form (2.5), classically the system (4.1) is the system with the nonlinear holomorphic supersymmetry of order n. Furthermore, the quantum shift of the parameters in the bosonic factors of the odd integrals (4.10) is similar to the shift in the factors of the holomorphic supercharges (2.14). However, the corresponding superpotential $W = \alpha x^{-1}$ does not satisfy the equation (2.10), and the form of the quantum analogues of the Hamiltonian and supercharges $S^\pm_{n,0}$ does not correspond to that of the nonlinear holomorphic supersymmetry. Note also that at $n = 2$ the quantum Hamiltonian $\hat{H}$ and the supercharges $\hat{S}^\pm_{n,0}$ given by equations (4.8), (4.10) and (4.11) coincide with the Hamiltonian and supercharges (3.3), (3.4) of the system with $W(x) = \left(\alpha - \frac{\hbar}{2}\right)x^{-1}$ and $\nu = 0$.

The structure of the quantum system (4.8)–(4.11) admits an interesting interpretation. Let us consider the system of a free planar spinning particle described by the Lagrangian

$$L = \frac{1}{2}\dot{x}_i^2 - \frac{\mathrm{i}}{2}\dot{\xi}_i\xi_i. \tag{4.12}$$

Being a free system, it is characterized by the integrals of motion p_i, $X_i = x_i - p_i t$ and ξ_i. Via the Poisson–Dirac brackets $\{x_i, p_j\} = \delta_{ij}$, $\{\xi_i, \xi_j\} = -\mathrm{i}\delta_{ij}$, $i, j = 1, 2$, they generate the space translations, the Galilei boosts and the supertranslations. Any function of p_i, X_i and ξ_i is also an integral of motion, and, in particular, the even quadratic functions

$$H = \frac{1}{2}p_i^2 \qquad K = \frac{1}{2}X_i^2 \qquad D = \frac{1}{2}X_i p_i \qquad \Sigma = -\frac{\mathrm{i}}{2}\epsilon_{jk}\xi_j\xi_k \tag{4.13}$$

$$J = L + \Sigma \tag{4.14}$$

where $L = \epsilon_{ij}X_i p_j$ are the integrals of motion generating the Lie algebra $so(1,2) \oplus u(1) \oplus u(1)$. Here the two terms $u(1)$ correspond to the spin Σ and the total angular momentum J, while the conformal symmetry $so(1,2)$ is generated by the Hamiltonian H, by the scale transformations generator D and by the generator of the special conformal transformations K.

The system (4.12) has more bosonic degrees of freedom than the system (4.1), and bosonic and fermionic degrees of freedom are not coupled in it. One can decrease the number of variables and couple appropriately the bosonic and fermionic degrees of freedom by introducing the constraint fixing the orbital motion:

$$\mathcal{J}_n - \alpha = 0 \qquad \mathcal{J}_n \equiv L + n\Sigma \tag{4.15}$$

where α is a real constant and $n \in \mathbb{N}$. Then the quantities having zero brackets with the constraint (4.15) are invariant with respect to the gauge transformations generated by it, and can be identified as observables of the system (4.12) supplemented with the constraint (4.15). In order to identify all the set of observables, we define the complex variables

$$X_\pm = \frac{1}{\sqrt{2}}(X_1 \pm \mathrm{i}X_2) \qquad P_\pm = \frac{1}{\sqrt{2}}(p_1 \pm \mathrm{i}p_2) \qquad \xi_\pm = \frac{1}{\sqrt{2}}(\xi_1 \pm \mathrm{i}\xi_2) \tag{4.16}$$

with nontrivial brackets $\{X_+, P_-\} = \{X_-, P_+\} = 1$, $\{\xi_+, \xi_-\} = -\mathrm{i}$, and find that in addition to the even observables

$$H = P_+P_- \qquad D = \frac{1}{2}(X_+P_- + P_+X_-) \qquad K = X_+X_- \qquad \Sigma = \xi_+\xi_-$$

and $\mathcal{J}_n$ given by equation (4.15) with $L = \mathrm{i}(X_+P_- - X_+P_-)$, there is a set of odd observables

$$S^+_{n,l} = 2^{n/2}\mathrm{i}^{n-l}P_-^{n-l}X_-^l\xi_+ \qquad S^-_{n,l} = \left(S^+_{n,l}\right)^*. \tag{4.17}$$

This set of even and odd observables forms on the constraint surface the Lie superalgebra $osp(2|2)$ ($n = 1$), or the nonlinear superconformal algebra $osp(2|2)_n$ ($n > 1$) given by equations (4.4)–(4.7). The free planar spinning particle system being reduced classically to the constraint surface reproduces the system (4.1). Moreover, the application of the Dirac quantization prescription applied to the system (4.12), (4.15) (first quantize and then reduce), reproduces correctly all the quantum corrections to the integrals of motion necessary for the preservation of the nonlinear superconformal symmetry at the quantum level (for details see [27]).

5. Discussion and outlook

We have seen that the quantization of the nonlinear supersymmetry faces the problem of the quantum anomaly. For some classes of the superpotentials, it is possible to find corrections to the integrals of motion which guarantee the preservation of the nonlinear supersymmetry at the quantum level. However, the origin of such corrections is not clear in a generic case.

The example of system (4.1) shows that the corrections can be understood as appearing due to the quantum Dirac reduction procedure applied to the appropriately chosen free extended system supplied with the constraint generating the necessary boson–fermion coupling. It was also observed in [13] that the nonlinear supersymmetry of non-holomorphic form [14] related to quasi-exactly solvable systems with sextic potential can be obtained via reduction of a planar system possessing nonlinear holomorphic supersymmetry. At the same time, it is known that a broad class of the quasi-exactly solvable systems, to which the systems with nonlinear supersymmetry are intimately related, can be obtained by the dimensional reduction of the corresponding extended systems [29]. Note that the same is true with respect to some integrable systems [30]. The nonlinear symmetries of a non-supersymmetric nature can also be obtained by the Hamiltonian reduction [1].

Therefore, it would be interesting to investigate the problem of the quantum anomaly associated with nonlinear supersymmetry within the general framework of reduction. This could be helpful, in particular, for establishing the nature of the universal quantum correction (3.5) for the Calogero-like systems with $n = 2$ supersymmetry. The idea of reduction could also be useful for clarifying the origin of the Dolan–Grady relations, on which the construction of the nonlinear holomorphic supersymmetry is based, as well as for its generalization for the case of many particle systems. In such a way one could try to answer the intriguing question on the possibility of realizing the nonlinear supersymmetry at the field level.

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